Matrix Rank

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Review

RREF

Definition

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

1. The leading entry in each non-zero row is 1.
2. Each leading 1 is the only non-zero entry in its columns.
3. The leading 1 in the second row or beyond is to the right of the leading 1 in the row just above.
4. Any row containing only 0's is at the bottom.

$$\underbrace{\begin{bmatrix} \overset{e_1}{1} & \overset{e_2}{0} & \overset{e_3}{0} & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}}_{\text{Reduced Echelon form}}$$

Number of non-zero rows = pivot columns

01

Row & Column Spaces

Row and Column Space

Definition

Let A be a $m \times n$ matrix:

- Then the ***column space*** of A is the collection of all linear combinations of its columns.
- The ***row space*** of A is the collection of all linear combinations of its rows.

Definition

Let A be a $m \times n$ matrix. Then the ***column space*** of A is C(A):

$$C(\mathrm{A}) \coloneqq \{ Ax : x \in \mathbb{R}^n \}$$

and the ***row space*** of A is:

$$R(\mathrm{A}) \coloneqq \{ y^T A : y \in \mathbb{R}^m \}$$

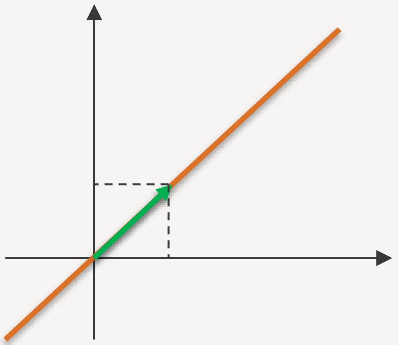
Note: We like vectors in column form so we can rewrite $R(A) = C(A^T)$

Row Space

- The row space of a matrix is the collection of all linear combinations of its rows: the row space is the span of rows.

$c_1[1 \quad 1] + c_2\,[0 \quad 0]$

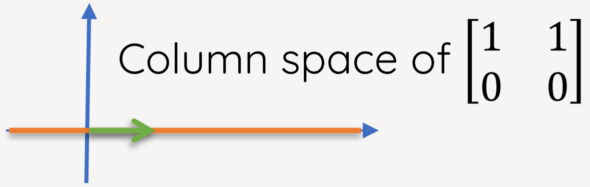
Row space of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$



- The elements of a row space are row vectors.
- For a $m \times n$ matrix, its row space is a subspace of (the row version of) $\mathbb{R}^n$. **Why?**

Column Space

- The column space of a matrix is the collection of all linear combinations of its columns.



$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- It is the span of columns, the image of the linear transformation carried out by the matrix.
- For a $m \times n$ matrix, its column space is a subspace of $\mathbb{R}^m$. Why?

Row-Equivalent

Theorem 1

If two matrices $A\ and\ B$ are row-equivalent, then their row spaces are the same.

Theorem 2

If two matrices $A\ and\ B$ are row-equivalent & B is in echelon form, the non-zero rows (pivot rows) of B form a basis for the row space of A as well as for that of B.

Pivot Columns

Theorem 3

The pivot columns of a matrix A form a basis for $C(A)$

$$\begin{bmatrix} 1 & b_{12} & 0 & b_{14} & 0 & b_{16} \\ 0 & 0 & 1 & b_{24} & 0 & b_{26} \\ 0 & 0 & 0 & 0 & 1 & b_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Lemma1: The pivot columns of A are linearly independent
Lemma 2. The pivot columns of A span the column space of A
From first lectures we know that “The span of the pivot columns is the same as the span of all the columns”

Row Space

- Elementary row operations do not alter the row space.

- Thus a matrix and its echelon form have the same row space. The pivot rows of an echelon form are linearly independent.

$$\begin{bmatrix} 1 & * & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The pivot rows of an echelon form span the row space of the original matrix. The dimension of the row space is given by the number of pivot rows.
- This dimension does not exceed the total row count.

Column Space

- Elementary row operations affect the column space.
- A matrix and its echelon form have different column spaces. However, since the row operations preserve the linear relations between columns, the columns of an echelon form and the original columns obey the same relations. The pivot columns of a reduced row-echelon form are ***linearly independent***.

$$\begin{bmatrix} 1 & * & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & \end{bmatrix}$$

Column Space

- Column space of a matrix is not same as the column space of row reduce version of matrix.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \quad , \quad B = RREF(A) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column Space

- Example

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 8 & 11 & 14 \\ 1 & 3 & 5 & 8 & 11 \\ 4 & 10 & 16 & 23 & 30 \end{pmatrix} \qquad B_{\text{rref}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 &= x_3 - x_5, \\ x_2 &= -2x_3 + 2x_5, \\ x_4 &= -2x_5. \end{aligned} \qquad \mathbf{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

The column space of B is 3−dimensional, and that a basis is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \\ 10 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ 8 \\ 23 \end{bmatrix} \right\}$

Note that we do not use the columns of B_{rref}! We use the columns of B.

Column Space

- The pivot columns of a reduced row-echelon form span its column space.
- The pivot columns of a matrix are linearly independent and span its column space.
- The dimension of the column space is given by the number of pivot columns.
- This dimension does not exceed the total column count.

02

Null Space

Lets Think!

- Consider this equation: $A_{m\times n}x_{n\times 1} = b_{m\times 1}$.
  Does all x s make a subspace in $\mathbb{R}^n$?

- Consider this equation: $A_{m\times n}x_{n\times 1} = 0_{m\times 1}$
  Does all x s make a subspace in $\mathbb{R}^n$?

Null Space

Definition

Let A be a $m \times n$ matrix. Then the ***null space or kernel*** of A is N(A):

$$N(\mathrm{A}) \coloneqq \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$$

Theorem 4

Consider $A_{m \times n}$ whose elements are from $set\ \mathcal{H}$ is a. The null space of matrix a ($\mathrm{N}(A)$) is subspace of $\mathcal{H}^n$. So it is a vector space!

Definition

Let A be a $m \times n$ matrix. Then null space or kernel of A^T is **Left Null Space of A**:

$$\mathrm{N}(\mathrm{A}^\mathrm{T}) = \{x \in \mathbb{R}^m : A^T x = \mathbf{0}\} = \{x \in \mathbb{R}^m : x^T A = \mathbf{0}^T\}$$

Null Space Basis

- Solve equation $Ax = \mathbf{0}$
- The spanning set of the solution form a basis for the null space.
- The number of free variables will indicate how many vectors are in the basis.

Example

Find a basis for N(A):

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 4 & -1 & 1 & -1 \\ 8 & -2 & 3 & -1 \end{bmatrix} \text{ reduces to } \begin{bmatrix} -4 & 1 & -3 & -4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

03

Column, Row Ranks & Nullity

Introduction

- Column space, row space, and null space are vector space.
- We can find basis for column space, row space, and null space.
- We can define the Dimension of column space, row space, and null space.

Column Rank

Definition

- We refer to a basis of C(A) consisting of columns of A as a column basis.
- We call dim(C(A)) the column rank of A.

ColRank(A)=dim(C(A))
= number of linear independent columns
= number of pivot columns

Row Rank

Definition

- We refer to a basis of R(A) consisting of rows of A as a row basis.
- We call dim(R(A)) the row rank of A.

$$\text{RowRank(A)} = \dim\big(\text{R(A)}\big) = \dim(C(A^T))$$

=number of linear independent rows

=number of non-zero rows in RREF

= number of pivot columns of A^T

Nullity

Definition

We call dim(N(A)) the nullity of A.

$\dim\big(\mathrm{N(A)}\big)$ =number of free variables

Example

If Columns of matrix A are linearly independent:

$$Nullity(A) \ = ?$$
$$ColRank(A) \ = ?$$

Practice

Example

- Basis of Row Space Matrix A
- Basis of Column Space Matrix A
- Basis of Null Space Matrix A
- $\dim(R(A))$
- $\dim(C(A))$
- $\dim(N(A))$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = pivot columns

04

Rank of Matrix

Rank-Nullity Theorem

Theorem 5

Let A be a $m \times n$ matrix:

$$Nullity(A) + ColRank(A) = n$$

Conclusion:

$$\{number\ of\ non-pivot\ columns\} + \{number\ of\ pivot\ columns\} = \{number\ of\ columns\}$$

ColRank=RowRank

Theorem 6

Let A be a $m \times n$ matrix:

- ❑ $Ax = 0 \quad \leftrightarrow \quad A^T Ax = 0$ *Why?*
- ❑ $\mathrm{ColRank}(\mathrm{A^T A}) = \mathrm{ColRank(A)}$ *Why?*
- ❑ $\dim(C(BD)) \le \dim(C(B))$ *or* $ColRank(BD) \le ColRank(B)$ *Why?*

Using the three above lemmas proof that:

- ❑ $\mathrm{ColRank(A)} = \mathrm{ColRank}(\mathrm{A^T})$

Then conlucde that $\mathbf{ColRank(A) = RowRank(A)}$

3. As a set of matrix-vector products.

$$C = AB = A\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{bmatrix}$$

Here the ith column of C is given by the matrix-vector product with the vector on the right, $c_i = Ab_i$

In general **Rank of matrix** is its column or row rank!

Rank-Nullity Theorem

Theorem 7

Let A be a $m \times n$ matrix:

$$Nullity(A) + Rank(A) = n$$

Conclusion:

$$\{number\ of\ non-pivot\ columns\} + \{number\ of\ pivot\ columns\} = \{number\ of\ columns\}$$

Rank of Matrices Multiplication

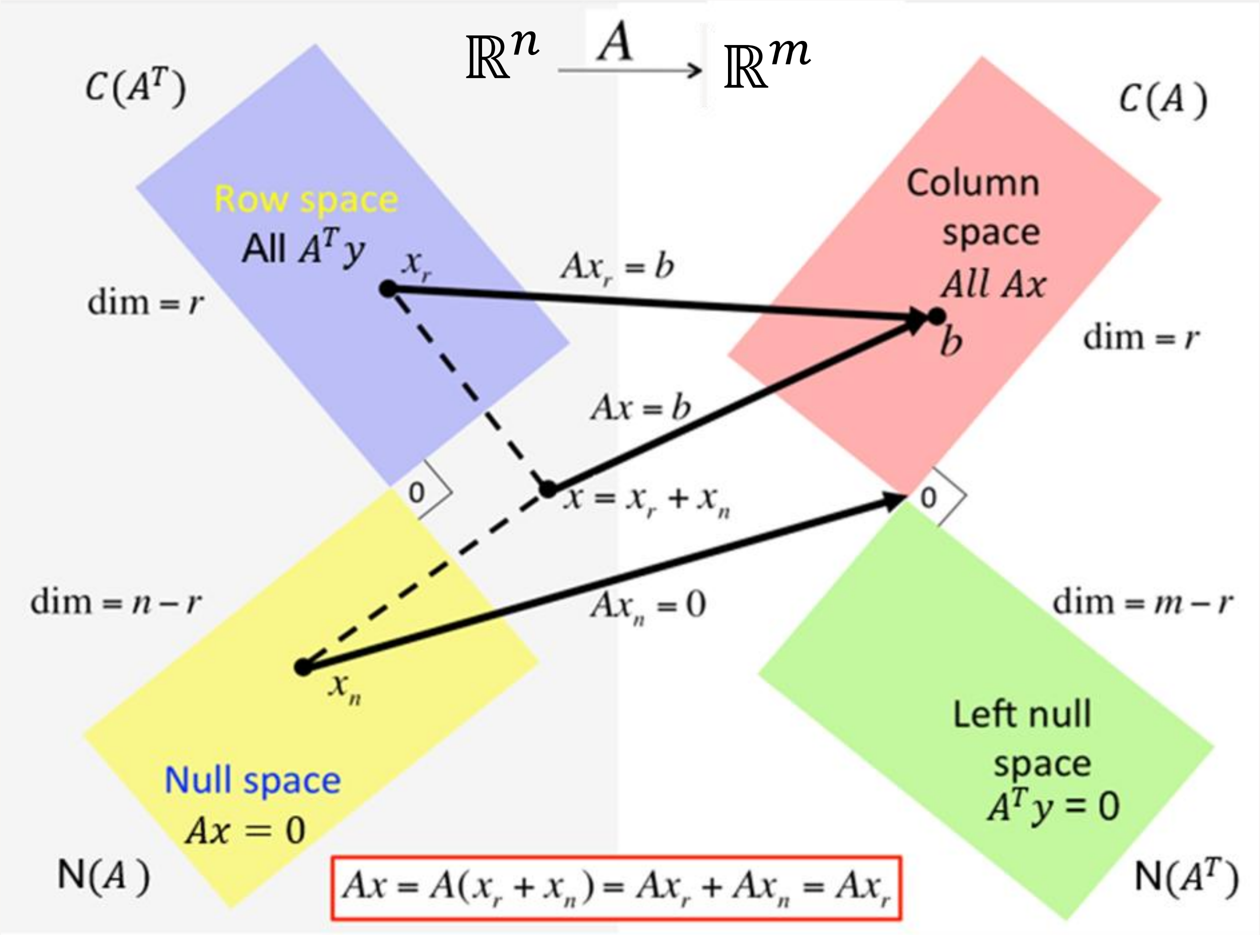
Theorem 8

$$rank(AB) \leq min\{rank(A), rank(B)\}$$

05

Four Fundamental Subspaces

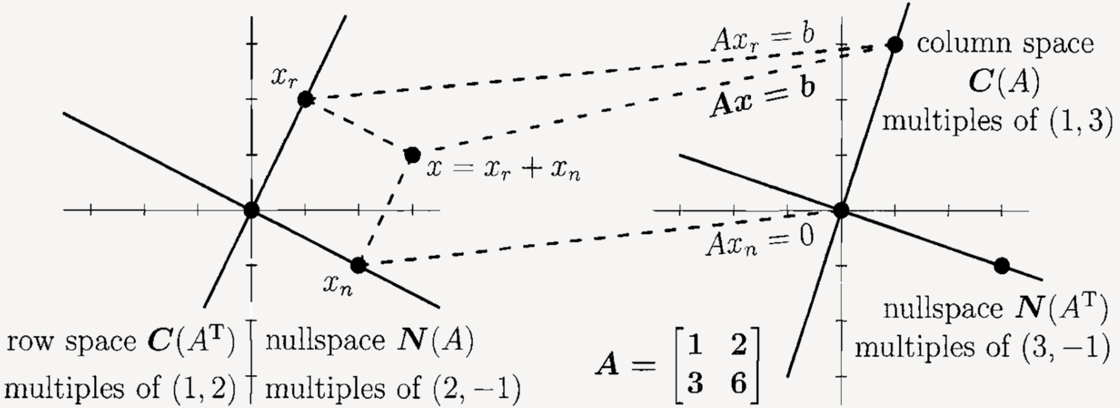
(of Matrix Space)


$\mathbb{R}^n$
A
$\mathbb{R}^m$
$C(A^T)$
$C(A)$
Row space
All $A^T y$
x_r
$Ax_r = b$
dim = r
Column space
All Ax
b
dim = r
Ax = b
0
$x = x_r + x_n$
0
dim = n − r
$Ax_n = 0$
dim = m − r
x_n
Left null space
$A^T y = 0$
Null space
$Ax = 0$
N(A)
$Ax = A(x_r + x_n) = Ax_r + Ax_n = Ax_r$
$N(A^T)$

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $m = n = 2$, and $r = 1$

- The column space has all multiples of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
- The row space contains all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
- The null space contains all multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
- The left null space contains all multiples of y = $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

The rows of A with coefficients -3 and 1 add to zero, $A^T y = 0$



Here the row space $C(A^T)$ is the multiples of (1,2), the null space $N(A)$ is the multiples of (2, -1), column space $C(A)$ is the multiples of (1,3) and left null space $N(A^T)$ is multiples of (3,-1). Also we can see that Row space is orthogonal to Null space and Column space is orthogonal to Left null space.

Summary of the Four Subspaces

- Let A be a $m \times n$ matrix with rank=r.

m, n, r	dim R(A)	dim N(A)	dim C(A)	dim $N(A^T)$	Solvability of $Ax = b$
m = n = r	r	0	r	0	Solvable, $x = A^{-1}b$ is unique solution
m > n = r	r	0	r	m – r	Solvable if $b \epsilon C(A)$
n > m = r	r	n – r	r	0	Solvable, infinite solutions $x = x_p + N(A)$
r < min(m,n)	r	n - r	r	m - r	Solvable only if $b \epsilon C(A)$ Infinite solutions

Resources

- Chapter 2 Part 9 & Chapter 4 Part 6, David C. Lay, Linear Algebra and Its Applications.
- Chapter 2 Part 4, Gilbert Strang, Linear Algebra and Its Applications.
- Chapter 4, Part 9, Bernard Kolman, Elementary Linear Algebra with Applications.